MATH 20D Spring 2023 Lecture 11.

Inhomogeneous Equations and the Method of Variation of Parameters

Outline

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- No homework due this week $\implies$ small adjustment to grading rubric. Homework now accounts for 22% of grade instead of 25%. Missing 3% has been added onto final exam.

Contents

Definition

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Theorem

*Suppose $y_p(t)$ is a fixed **particular solution** to the ODE (2). If $y_{\mathrm{sol}}(t)$ is any solution to (2) then there exists a complementary solution $y_c(t)$ such that $y_{\mathrm{sol}}(t) = y_p(t) + y_c(t)$.*

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- Show that $y_p(t) = \frac{1}{3}e^t$ is a particular solution to equation (3).
- Using the method of integrating factors, show that if $y_{\text{sol}}(t)$ is a solution to (3) then there exists a complementary solution $y_c(t)$ such that

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- The above example says that $y'(t) + 2y(t) = e^t$ admits a general solution of the form

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t) = \frac{1}{3}e^t$ and $y_h(t)$ is a general solution of $y'(t) + 2y(t) = 0.$

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we can construct a general solution by writting $y(t) = y_p(t) + y_h(t)$ where $y_h(t)$ is a general solution to the **homogeneous equation** (4).

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Question

Given an inhomogeneous ODE's of the form
$$ay''(t) + by'(t) + cy(t) = f(t) \tag{5}$$
is there a method for constructing a particular solution to (5)?

Theorem

Suppose $a \neq 0$, b, and c are constant we are given an inhomogeneous ODE

$$ay''(t) + by'(t) + cy(t) = f(t). \tag{6}$$

and let $y_1(t)$ and $y_2(t)$ be linearly independent solutions to the corresponding homogeneous equation.

- *If $v_1(t)$ and $v_2(t)$ are functions satisfying the system of equations*

$$\begin{cases} y_1(t)v_1'(t) + y_2(t)v_2'(t) = 0 \\ y_1'(t)v_1'(t) + y_2'(t)v_2'(t) = f(t)/a \end{cases}$$

then $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ gives a ***particular*** *solution to the (6).*

Example

Using the method of variation of parameters find a particular solution to $y'' + 9y = 1$. Using your answer give a general solution to $y'' + 9y = 1$.